Physics 309

Solving the Harmonic Oscillator Schroedinger Equation I

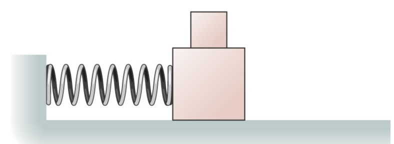
1. The general solution to the classical harmonic oscillator is $x(t) = A\sin(\omega_0 t + \delta)$. Get an expression for the period of the motion (the time to make one complete oscillation) in terms of the parameters of the general solution. How is this result related to the frequency?

2. A $50-g$ mass connected to a spring of force constant 35 N/m oscillates on a horizontal, frictionless surface with an amplitude of 4.0 cm. Find (a) the total energy of the system and (b) the speed of the mass when the displacement is 1.0 cm. When the displacement is 3.0 cm, find (c) the kinetic energy and (d) the potential energy.

3. A car with bad shock absorbers bounces up and down with a period of 1.5 s after hitting a bump. The car has a mass of 1500 kg and is supported by four springs of equal force constant k. What is k?

4. A mass m is oscillating freely on a vertical spring. When $m = 0.810$ kg, the period is 0.910 s. An unknown mass on the same spring has a period of 1.16 s. What is the spring constant k and the unknown mass.

5. A 1.0 kg cube oscillates horizontally on the end of a spring like the one shown here. The extreme displacement of the mass as it oscillates is 0.10 m and its period of oscillation is 0.50 s. What is the spring constant? After 27 periods, the cube comes to rest. What is the energy dissipated by friction?



6. In solving the Schroedinger equation for the harmonic oscillator potential we rewrote the Schroedinger equation in the form

$$\frac{d^2\phi}{d\xi^2} + \left(\frac{\alpha}{\beta^2} - \xi^2\right)\phi = 0$$

where $\xi = \beta x$, $\alpha = 2mE/\hbar^2$ and $\beta = \sqrt{m\omega_0/\hbar}$. What is the asymptotic form of this differential equation? In other words, what does it look like for large ξ? Show the asymptotic solution is

$$|\phi_{asymp}\rangle = A_{asymp}e^{-\xi^2/2} + B_{asymp}e^{\xi^2/2} \quad .$$

7. Once we established the form of the asymptotic solution of the harmonic oscillator problem we made the initial guess that the wave function will be of the form

$$|\phi(\xi)\rangle = Ae^{-\xi^2/2}H(\xi)$$

where $\xi = \beta x$, $H(\xi)$ is some, as-yet-to-be-determined function, and A is a normalization constant. This guess was made in the hope of ensuring the finiteness of the wave function far outside the range of the potential. Starting from this form of the wave function and the Schroedinger equation, show the new differential equation we must solve is

$$\frac{d^2H(\xi)}{d\xi^2} - 2\xi\frac{dH(\xi)}{d\xi} + \left(\frac{\alpha}{\beta^2} - 1\right) H(\xi) = 0$$

where

$$\alpha = \frac{2mE}{\hbar^2} \qquad \text{and} \qquad \beta^2 = \frac{2\pi m\nu_0}{\hbar} = \frac{m\omega_0}{\hbar} \qquad .$$

8. Recall our old friends, Newton's Second Law, $\vec{F} = m\vec{a}$ and Hooke's Law, $|\vec{F}| = -Kx$ which can be combined to form a differential equation

$$m\frac{d^2x}{dt^2} = -Kx \qquad \text{or} \qquad \frac{d^2x}{dt^2} = -\omega_0^2 x$$

where $\omega_0 = \sqrt{\frac{K}{m}}$. The solutions of this equation have, of course, been known to us since our earliest childhood, but now solve this differential equation using the Method of Frobenius and make the appropriate choices of the leading coefficients to obtain those well known solutions.

9. If one makes an astute choice of units then the first two harmonic oscillator wave functions can be written as:

$$\psi_0 = A_0 e^{-\xi^2/2} \qquad \textit{and} \qquad \psi_1 = A_1 \xi e^{-\xi^2/2}$$

where

$$\xi = \beta x \qquad , \qquad \beta^2 = \frac{m\omega_0}{\hbar}.$$

Find the normalization constants, A_0 and A_1.

10. The energy eigenvalues of a molecule indicate the molecule is a one-dimensional harmonic oscillator. In going from the second excited state to the first excited state, it emits a photon of energy $h\nu = 0.1\ eV$. Assuming that the oscillating portion of the molecule is a proton, calculate the probability that a proton in the first excited state is at a distance from the origin that would be forbidden to it by classical mechanics. You may have difficulty performing the integration necessary for the final answer. In that case, express that answer in terms of the unsolved integral and propose some method for calculating it. A special prize awaits anyone who can solve the integral.

11. After we first found the series solution to the harmonic oscillator potential we discovered a problem. When subjected to the ratio test our series converged to $2/n$ for large n, where n is the term in the series. The claim was then made that this behavior is consistent with the behavior of the function e^{u^2} at large u. This feature implies the wave function blows up at large distances from the equilibrium point. Show that the Taylor series expansion of the function e^{u^2} about the origin has the following form.

$$e^{u^2} = 1 + u^2 + \frac{u^4}{2!} + \cdots + \frac{u^n}{(n/2)!} + \cdots$$

Next, use the ratio test to show this series converges to $2/n$ for large u.

12. Recall again our old friend, Newton's Second Law, $\vec{F} = m\vec{a}$ and perhaps a new one in the drag force equation $F_d = -bv$ which can be combined to form a differential equation in the velocity for an object falling straight down

$$m\frac{dv}{dt} = bv - mg \qquad \text{or} \qquad \frac{dv}{dt} - \frac{b}{m}v + g = 0$$

where b is a parameter describing the drag force, m is the mass and g is the acceleration of gravity. Solve this differential equation using the Method of Frobenius (the power series method) and generate the recursion relationship that relates different coefficients to one another.