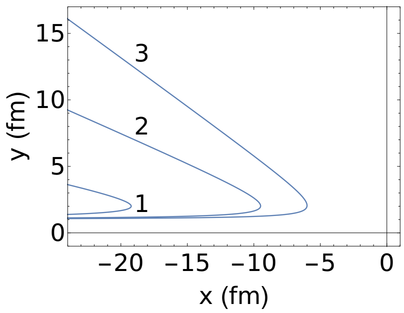
Physics 303 Final

I pledge that I have neither given nor received unauthorized assistance during the completion of this work.

Signature ______________________________

Questions (5 pts. apiece) Answer questions 1-8 in complete, well-written sentences WITHIN the spaces provided.

1. The figure shows some trajectories for Rutherford scattering. What is the impact parameter for each trajectory? What is the order of the beam energies from lowest to highest? Explain.



2. To measure the distance from star to planet r_{sp} for an exoplanet we use Kepler's Law

$$r_{sp}^3 = \frac{GT^2M_s}{4\pi^2}$$

where G is the Gravitational constant, T is the measured period of oscillation, and M_s is the mass of the star. How do we get the value of M_s?

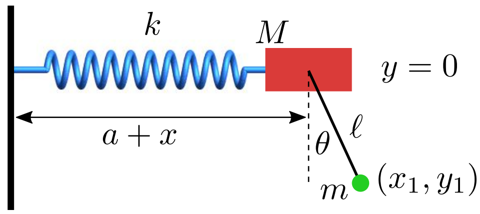
3. For an object (like Lt. Chisov) falling through the atmosphere, why does it reach a maximum speed and stop accelerating (assuming the configuration of the object does not change)? Explain.

4. For the oscillating biosensor you are designing for Homeland Security should it be an underdamped or overdamped oscillator? Explain.

5. The Lagrangian for a pendulum of mass m attached to another mass M on a spring moving horizontally (see the figure) is

$$\mathscr{L} = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}\ell^2\dot{\theta}^2 + m\ell\cos\theta\dot{x}\dot{\theta} - \frac{1}{2}kx^2 + mg\ell\cos\theta$$

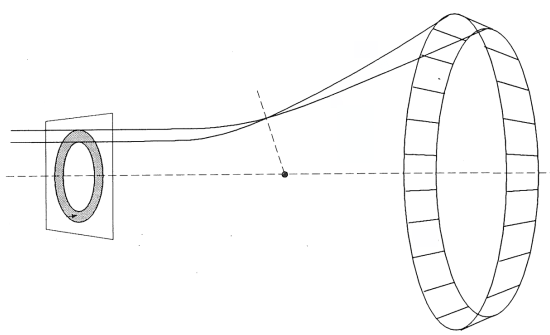
where k is a spring constant. What is the x component of the momentum?



6. Can we integrate the equation below? Why or why not?

$$\frac{dv}{dt} = \cot\theta$$

7. In the figure below showing the interaction of two nuclei in a scattering experiment, clearly label the impact parameter b, differential impact parameter db, scattering angle θ_s, differential scattering angle $d\theta_s$, solid angle $d\Omega$, and differential area dA.



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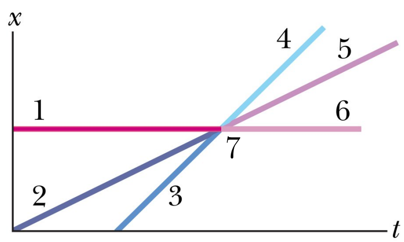
8. Two bodies undergo an elastic, one-dimensional collision along an x axis. The figure below is a graph of position versus time for those bodies and for their center of mass (CM). (a) Were both bodies initially moving, or was one initially stationary? Which line segment corresponds to the motion of the CM (b) before the collision and (c) after the collision? (d) Is the mass of the body that was moving faster before the collision greater than, less than, or equal to that of the other body? Explain your results.

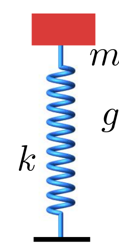


Problems (1-6). Clearly show all reasoning for full credit. Use a separate sheet to show your work.

1. 8 pts. A spaceship, moving away from Earth at a speed of $v_a = 0.9c$, reports back by transmitting at a frequency (measured in the spaceship frame) of $f = 100\ MHz$. To what frequency must Earth receivers be tuned to receive the report?

2. 8 pts. A massless spring of rest length ℓ and spring constant k has a mass m attached to one end. The system is set on a table with the mass vertically above the spring as shown.

   1. What is the new equilibrium height of the mass above the table?
   2. The spring is compressed a distance below the new equilibrium height and released. What is the new equation of motion of the mass assuming the free end of the spring remains in contact with the table.



3. 10 pts. Consider the equation for free fall with air friction

$$F = m\frac{dv}{dt} = -mg + cv^2$$

where m is the mass, g is the acceleration of gravity, and c is the drag coefficient where the terminal velocity $v_t = \sqrt{mg/c}$. Integrate this equation of motion to obtain the position as a function of velocity $x(v)$. The position here is the distance fallen by a sky diver in free fall. Get $x(v)$ including any constants of integration assuming the initial velocity is zero and the initial position is at the origin.

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4. 10 pts. Consider a planet orbiting a star in a plane where the mass of the planet m_p is much less than the mass of the star M_s so $m_p \ll M_s$. What is the Lagrangian for the system? If the planet follows a circular orbit, then show, starting from the Lagrangian

$$r_{sp}^3 = \frac{GT^2 M_s}{4\pi^2}$$

where T is the period of the planet's orbit and r_{sp} is the star-planet distance.

5. 12 pts. Two masses m_1 and m_2 are connected by a spring of rest length ℓ and spring constant k. The system glides without friction on a horizontal surface in the direction of the spring's length.

   1. What is the Lagrangian for the system?
   2. Starting from the Lagrangian, what are the equations of motion of the system?
   3. Find the normal modes by assuming oscillatory solutions for each coordinate. What are the corresponding frequencies?

6. 12 pts. The color force that binds quarks together in protons, neutrons, and atomic nuclei has been successfully modeled using the potential energy

$$V(r) = -\frac{k_1}{r} + k_2 r \qquad k_1 > 0,\ k_2 > 0$$

where r is the quark-quark separation.

   1. What is the force?
   2. What is the angular momentum L of a bound quark with mass m on a circular orbit of radius a in terms of k_1, k_2, a, and m?
   3. What is the total energy E for motion of a bound quark with mass m on a circular orbit of radius a in terms of k_1, k_2, a, and m?
   4. What is the period of circular motion in terms of the same quantities?

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Physics 303 Final Equation Sheet

$$\vec{F} = m\vec{a} = \dot{\vec{p}} = -\frac{dV}{dr}\hat{r} \quad \vec{F}_G = -\frac{Gm_1m_2}{r^2}\hat{r} \quad \vec{F}_C = \frac{kq_1q_2}{r^2}\hat{r} \quad \vec{F}_g = -mg\hat{y} \quad \vec{F}_s = -kr\hat{r} \quad |\vec{F}_{cent}| = m\frac{v^2}{r}$$

$$\vec{F}_f = -bv\hat{v} \quad \vec{F}_f = -cv^2\hat{v} \quad \frac{df(y)}{dx} = \frac{df(y)}{dy}\frac{dy}{dx} \quad \int \frac{df}{dx} = \int df \quad f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\ddot{y} + A\dot{y} + By = 0 \Rightarrow y = Ce^{\lambda t} \quad \ddot{y} + \omega_0^2 y = 0 \Rightarrow y = A\sin(\omega_0 t + \phi) = \alpha_1 e^{i\omega_0 t} + \alpha_2 e^{-i\omega_0 t}$$

$$\ddot{y} + \omega_0^2 y = \omega_0^2 l \Rightarrow y = C + A\sin(\omega_0 t + \phi) \quad \text{or} \quad y = C' + A\sin\omega_0 t + B\cos\omega_0 t$$

$$\omega_0^2 = \frac{k}{m} \quad \omega_0^2 = \frac{g}{l} \quad k = \left.\frac{d^2V(x)}{dx^2}\right|_{x=x_e} \text{(small oscillations)}$$

$$V = -\int_{x_s}^{x}\vec{F}(\vec{r}\,')\cdot d\vec{r}\,' \quad E = K+V \quad V_s = \frac{kx^2}{2} \quad V_g = mgy \quad V_G = -\frac{Gm_1m_2}{r} \quad V_C = \frac{kq_1q_2}{r} = \frac{Z_1Z_2e^2}{r}$$

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad \mathscr{L} = K-V \quad \frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}}\right) - \frac{\partial\mathscr{L}}{\partial q} = 0 \quad p_q = \frac{\partial\mathscr{L}}{\partial\dot{q}} \quad (\mathbf{A}-\lambda\mathbf{1})\vec{C} = 0$$

$$\vec{L} = \vec{r}\times\vec{p} \quad \vec{N} = \frac{d\vec{L}}{dt} \quad L = \mu r^2\dot{\theta} \quad V_{cent} = \frac{L^2}{2\mu r^2} \quad \vec{p} = m\vec{v} \quad \vec{p}_i = \vec{p}_f \quad K_i = K_f \quad e = \frac{|\vec{v}_{2f} - \vec{v}_{1f}|}{|\vec{v}_{2i} - \vec{v}_{1i}|}$$

$$\frac{1}{r} = \frac{\mu\alpha}{L^2}(1 + \epsilon\cos(\theta - \theta_0)) \quad \epsilon = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}} \quad \sin\left(\frac{\theta_s}{2}\right) = \frac{1}{\epsilon} \qquad \frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E_{cm}}\right)^2 \frac{1}{\sin^4\left(\frac{\theta_s}{2}\right)}$$

$$\frac{dN_s}{dt} = \frac{d\sigma}{d\Omega}n_{tgt}d\Omega \quad \alpha = Gm_1m_2 \text{ or } -e^2Z_1Z_2 \quad E_{cm} = \frac{m_{tgt}}{m_{tgt} + m_{beam}}E_{lab} \quad V_{eff}(r) = \frac{L^2}{2\mu r^2} + V(r)$$

$$\mu = \frac{m_1m_2}{m_1 + m_2} \quad \vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \vec{0} \quad v_T = r\dot{\theta} = \frac{2\pi r_{sp}}{T} \approx \sqrt{\frac{GM_s}{r_{sp}}} = v_p \quad r_{sp} = \left[\frac{GT^2M_s}{4\pi^2}\right]^{1/3}$$

$$\mu\ddot{r}_{sp} - \mu r_{sp}\dot{\theta}^2 + \frac{\alpha}{r_{sp}} = 0 \quad r_s = \frac{m_p}{m_p + m_s} \quad \lambda_\pm = \sqrt{\frac{1 \mp v_s/c}{1 \pm v_s/c}}\,\lambda_0 \quad c = \frac{\lambda}{T} = \lambda f$$

More Equations, Conversions, and Constants

Speed of light (c)	$3.0 \times 10^8\ m/s$	g	$9.8\ m/s^2$
Gravitation constant (G)	$6.67 \times 10^{-11}\ N-m^2/kg^2$	Earth's radius	$6.37 \times 10^6\ m$
Coulomb constant (k_e)	$8.99 \times 10^9 \frac{N-m^2}{C^2}$	Earth's mass	$5.97 \times 10^{24}\ kg$
Elementary charge (e)	$1.60 \times 10^{-19}\ C$	Proton/Neutron mass	$1.67 \times 10^{-27}\ kg$
Planck's constant (h)	$6.626 \times 10^{-34}\ J-s$	Proton/Neutron mass	$932 \times 10^6\ eV/c^2$
Permittivity constant (ϵ_0)	$8.85 \times 10^{-12} \frac{kg^2}{N-m^2}$	Electron mass	$9.11 \times 10^{-31}\ kg$
Permeability constant (μ_0)	$4\pi \times 10^{-7} N/A^2$	Electron mass	$0.55 \times 10^6\ MeV/c^2$
1 MeV	$10^6\ eV$	1.0 eV	$1.6 \times 10^{-19}\ J$
1 kg	$931.5\ MeV/c^2$	1 u	$1.67 \times 10^{-27}\ kg$
$\hbar c$	$197\ MeV-fm$	e^2	$\frac{hbarc}{137}$

$$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) \qquad \int \tanh x dx = \ln\left[\cosh x\right] \qquad \int \coth x dx = \ln[\sinh x]$$

$$\frac{d}{dx}\tan x = \frac{1}{\cos^2 x} \qquad \int \frac{1}{\sqrt{x^2+a^2}}dx = \sinh^{-1}\frac{x}{a} \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\ln ax = \frac{1}{x} \qquad \frac{d}{dx}\cot x = -\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\int \sqrt{a^2-x^2}dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \qquad \int \sqrt{x^2-a^2}dx = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\left[x+\sqrt{x^2-a^2}\right]$$

$$\int \tanh^2(x)dx = x - \tanh x \qquad \int \tanh^3(x)dx = \ln\left[\cosh x\right] + \frac{\operatorname{sech}^2(x)}{2}$$

$$\int \sqrt{\tanh x}dx = -\tan^{-1}\left[\sqrt{\tanh x}\right] - \frac{1}{2}\ln\left[1-\sqrt{\tanh x}\right] + \frac{1}{2}\ln\left[1+\sqrt{\tanh x}\right]$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x \qquad \frac{d}{dx}\sinh x = \cosh x$$

$$\int \frac{1}{\sqrt{ar^4+br^3-r^2}}dr = \frac{r\sqrt{-1+br+ar^2}\arctan\left(\frac{-2+br}{2\sqrt{-1+br+ar^2}}\right)}{\sqrt{r^2(-1+br+ar^2)}}$$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx = -\frac{1}{\sqrt{-a}}\arcsin\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right)$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Pythagorean identities :

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Reciprocal identities :

$\csc x = \dfrac{1}{\sin x}$

$\sec x = \dfrac{1}{\cos x}$

$\cot x = \dfrac{1}{\tan x}$

Even - odd identities :

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

Product to sum formulas :

$\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

$\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$

$\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$

Sum to product :

$\sin x \pm \sin y = 2\sin\left(\dfrac{x \pm y}{2}\right)\cos\left(\dfrac{x \mp y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

Double - angle formulas :

$\sin 2\theta = 2 \cdot \sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Co - function identities :

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$

$\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

$\csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

Periodicity identities :

$\sin(x \pm 2\pi) = \sin x$

$\cos(x \pm 2\pi) = \cos x$

$\tan(x \pm \pi) = \tan x$

$\cot(x \pm \pi) = \cot x$

$\sec(x \pm 2\pi) = \sec x$

$\csc(x \pm 2\pi) = \csc x$

Sum and difference formulas :

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Half - angle formulas :

$\sin\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1-\cos x}{2}}$

$\cos\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1+\cos x}{2}}$

$\tan\left(\dfrac{x}{2}\right) = \dfrac{(1-\cos x)}{\sin x}$

Law of sines :

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of cosines :

$a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$

Area of triangle :

$\frac{1}{2}ab\sin C$

$\sqrt{s(s-a)(s-b)(s-c)}$,

where $s = \frac{1}{2}(a+b+c)$

- Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} = \frac{1 - e^{-2x}}{2e^{-x}}.$$

- Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x} = \frac{1 + e^{-2x}}{2e^{-x}}.$$

- Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \\ = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

- Hyperbolic cotangent: $x \neq 0$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \\ = \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{1 + e^{-2x}}{1 - e^{-2x}}$$

- Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} = \\ = \frac{2e^x}{e^{2x} + 1} = \frac{2e^{-x}}{1 + e^{-2x}}$$

- Hyperbolic cosecant: $x \neq 0$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}} = \\ = \frac{2e^x}{e^{2x} - 1} = \frac{2e^{-x}}{1 - e^{-2x}}$$

$$\begin{aligned} \operatorname{arsinh}(x) &= \ln\left(x + \sqrt{x^2 + 1}\right) \\ \operatorname{arcosh}(x) &= \ln\left(x + \sqrt{x^2 - 1}\right); x \geq 1 \\ \operatorname{artanh}(x) &= \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right); |x| < 1 \\ \operatorname{arcoth}(x) &= \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right); |x| > 1 \end{aligned}$$

Odd and even functions:

$$\begin{aligned} \sinh(-x) &= -\sinh x \\ \cosh(-x) &= \cosh x \end{aligned}$$

Hence:

$$\begin{aligned} \tanh(-x) &= -\tanh x \\ \coth(-x) &= -\coth x \\ \operatorname{sech}(-x) &= \operatorname{sech} x \\ \operatorname{csch}(-x) &= -\operatorname{csch} x \end{aligned}$$