Chapter 1

ONE-DIMENSIONAL MOTION

The formulation of classical mechanics represents a giant milestone in our intellectual and technological history, as the first mathematical abstraction of physical theory from empirical observation. This crowning achievement is rightly accorded to Isaac Newton (1642–1727), who modestly acknowledged that if he had seen further than others, "it is by standing upon the shoulders of Giants." However, the great physicist Pièrre Simon Laplace characterized Newton's work as the supreme exhibition of individual intellectual effort in the history of the human race.

Newton translated the interpretation of various physical observations into a compact mathematical theory. Three centuries of experience indicate that mechanical behavior in the everyday domain can be understood from Newton's theory. His simple hypotheses are now elevated to the exalted status of laws, and these are our point of embarkation into the subject.

1.1 Newtonian Theory

The Newtonian theory of mechanics is customarily stated in three laws. According to the first law, a particle continues in uniform motion (*i.e.*, in a straight line at constant velocity) unless a force acts on it. The first law is a fundamental observation that physics is simpler when viewed from a certain kind of coordinate system, called an *inertial frame*. One cannot define an inertial frame except by saying that it is a frame in which Newton's laws hold. However, once one finds (or imagines) such a frame, all other frames which move with respect to it at constant velocity, with no rotation, are also inertial frames. A coordinate system fixed on the surface of the earth is not an inertial frame because of the acceleration due to the rotation of the earth and the earth's motion around the sun. Nevertheless, for many purposes it is an adequate approximation to regard a coordinate frame fixed on the earth's surface as an inertial frame. Indeed, Newton himself discovered nature's true laws while riding on the earth!

The essence of Newton's theory is the second law, which states that *the time rate of change of momentum of a body is equal to the force acting on the particle.* For motion in one dimension, the second law is

$$F = \frac{dp}{dt} \tag{1.1}$$

where the momentum p is given by the product of (mass) $\times$ (velocity) for the particle

$$p = mv \tag{1.2}$$

The second law provides a definition of force. It is useful because experience has shown that the force on a body is related in a quantitative way to the presence of other bodies in its vicinity. Further, in many circumstances it is found that the force on a body can be expressed as a function of x, v, and t, and so (1.1) becomes

$$F(x, v, t) = \frac{dp}{dt} = m\frac{d^2x}{dt^2} \tag{1.3}$$

This differential equation is called the *equation of motion.* Here m is assumed to be constant. For the remainder of this book we use Newton's notation $\dot{x} = dx/dt$; $\ddot{x} = d^2x/dt^2$. Newton's second law is then

$$F(x, \dot{x}, t) = m\ddot{x} = ma \tag{1.4}$$

where $a = \ddot{x}$ is the acceleration. In the special case $F = 0$, integration of (1.1) gives $p =$ constant in accordance with the first law.

While Newton's laws apply to any situation in which one can specify the force, very few interesting physical problems lead to force laws amenable to simple mathematical solution. The fundamental force laws of gravitation and electromagnetism do have simple forms for which the second law of motion can often be solved exactly. The use of approximate empirical forms to approximate the true force laws of physical situations involving frictional and drag forces is one of the arts that will be taught in this book. However, in this modern age of computers, one can handle arbitrary force laws by the brute-force method of numerical integration.

The third law states that if body A experiences a force due to body B, then B experiences an equal but opposite force due to A. (One speaks of this as the force between the two bodies.) As a consequence, the rates of change of the momenta of particles A and B are equal but opposite, and therefore the total $p_A + p_B$ is constant. This law is extremely useful, for instance in the treatment of rigid-body motion, but its range of applicability is not as universal as the first two laws. The third law breaks down when the interaction between the particles is electromagnetic, because the electromagnetic field carries momentum.

It is a remarkable fact that macroscopic phenomena can be explained by such a simple set of mathematical laws. As we shall see, the mathematical solutions to some problems can be complex; nevertheless, the physical basis is just (1.1). Of course, there is still a great deal of physics to put into (1.1), namely, the laws of force for specific kinds of interactions.

1.2 Interactions

Using the planetary orbit data analysis by Kepler, Newton was able to show that all known planetary orbits could be accounted for by the following force law

$$F = -\frac{GM_1M_2}{r^2} \tag{1.5}$$

This states that force between masses M_1 and M_2 is proportional to the masses and inversely proportional to the square of the distance between them. The negative sign in (1.5) denotes an attractive force between the masses. The force acts along the line between the two masses and thus for non-rotational motion the problem is effectively one-dimensional. Newton proposed that this gravitational law was universal, the same force law applying between us and the earth as between celestial bodies (and more generally between any two masses). The universality of the gravitational law can be verified, and the proportionality constant G determined, by delicate experimental measurements of the force between masses in the laboratory. The value of G is

$$G = 6.672 \times 10^{-11}\ \mathrm{m^3/(kg\,s)^2} \tag{1.6}$$

The dominant gravitational force on an object located on the surface of the earth is the attraction to the earth. The gravitational force between two spherically symmetric bodies is as if all the mass of each body were concentrated at its center, as Newton proved. We will give a proof of this assertion in Chapter 8. The earth is very nearly spherical so we can use the force law of (1.5). Thus for an object of mass m on the surface of earth, the force is

$$F = -m\frac{M_E G}{R_E^2} = -mg \tag{1.7}$$

where g is the gravitational acceleration,

$$g \simeq 9.8 \text{ m/s}^2 \tag{1.8}$$

Using the measured value of $R_E = 6{,}371$ km along with the measured values of g and G as given above, we may use (1.7) to deduce the mass of the earth to be

$$M_E = 5.97 \times 10^{24} \text{ kg} \tag{1.9}$$

Since the earth's radius is large, the gravitational force of an object anywhere in the biosphere is given to good accuracy by (1.7); even at the top of the atmosphere ($\approx$ 200 km up) the force has decreased by less than 10% from its value at the surface of the earth. Consequently, in many applications on earth, we can neglect the variation of the gravitational force with position.

The static Coulomb force between two charges e_1 and e_2 is similar in form to the gravitational-force law of (1.5),

$$F = k\frac{e_1 e_2}{r^2} \tag{1.10}$$

This force is attractive if the charges are of opposite sign and repulsive if the charges are of the same sign. The constant k depends on the system of electrical units; in *SI* units, $k = (4\pi\epsilon_0)^{-1} \simeq 9 \times 10^9 \text{N-m}^2/\text{C}^2$.

Another force with a wide range of application is the spring force or Hooke's law, which is expressed as

$$F = -kx \tag{1.11}$$

with $k > 0$. Here k is a spring constant which is dependent on the properties of the spring and x is the extension of the spring from its relaxed position. This particular force law is a very good approximation in many physical situations (*e.g.*, the stretching or bending of materials) which are initially in equilibrium.

Frictional forces prevent or damp motions. The static frictional force between two solid surfaces is

$$|F| \leq \mu_s N \tag{1.12}$$

The force F acts to prevent sliding motion. N is the perpendicular force (normal force) holding the surfaces together, and μ_s is a material-dependent coefficient. Equation (1.12) is an *approximate* formula for frictional forces which has been deduced from empirical observations. The frictional force which retards the motion of sliding objects is given by

$$F = \mu_k N \tag{1.13}$$

It is observed that this force is nearly independent of the velocity of the motion for velocities which are neither too small (where there is molecular adhesion) nor too large (where frictional heating becomes important). For a given pair of surfaces, the coefficient of kinetic friction μ_k is less than the coefficient of static friction μ_s.

Frictional laws to describe the motion of a solid through a fluid or a gas are often complicated by such effects as turbulence. However, for sufficiently small velocities, the approximate form

$$F = -bv \tag{1.14}$$

where b is a constant, holds. The drag coefficient b in (1.14) is proportional to the fluid viscosity. For a sphere of radius a moving slowly through a fluid of viscosity η the Stokes law of resistance is calculated to be

$$b_{\text{sphere}} = 6\pi a\eta \tag{1.15}$$

At higher, but still subsonic velocities, the drag law is

$$F = -cv^2 \tag{1.16}$$

For instance, the drag force on an airplane is remarkably well represented by a constant times the square of the velocity. The drag coefficient c for a body of cross-sectional area S moving through a fluid of density ρ is given by

$$c = \tfrac{1}{2}C_D S\rho \tag{1.17}$$

where C_D is a dimensionless factor related to the geometry of the body (about 0.4 for a sphere).

Externally imposed forces can take on a variety of forms. Of those depending explicitly on time, sinusoidally oscillating forces like

$$F = F_0 \cos \omega t \tag{1.18}$$

are frequently encountered in physical situations.

In a general case the forces can be position-, velocity-, and time-dependent,

$$F = F(x, v, t) \tag{1.19}$$

Among the most interesting and easily solved examples are those in which the forces depend on only one of the above three variables, as illustrated by the examples in the following three sections.

1.3 The Drag Racer: Frictional Force

A number of interesting engineering-type problems can be solved from straightforward application of Newton's laws. As an illustration, suppose we consider a drag racer that can achieve maximum possible acceleration when starting from rest. The external forces on the racer which must be taken into account are (1) gravity, (2) the normal forces supporting the racer at the wheels, and (3) the frictional forces which oppose the rotation of the powered rear wheels. A sketch indicating the various external forces is given in Fig. 1-1.

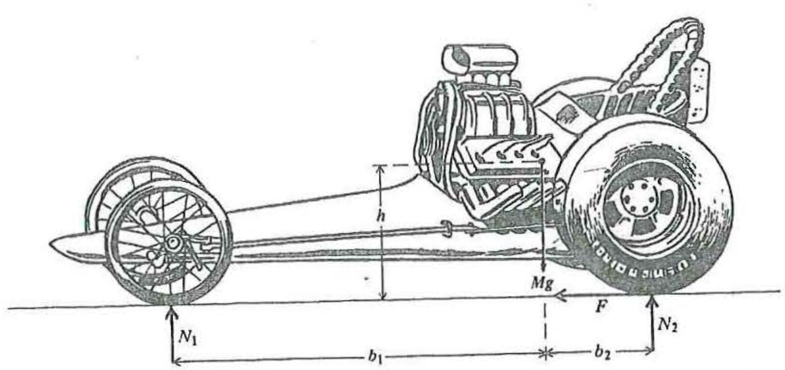


FIGURE 1-1. Forces on a drag racer.

Since the racer is in vertical equilibrium, the sum of the external vertical forces must vanish,

$$N_1 + N_2 - Mg = 0 \tag{1.20}$$

Both N_1 and N_2 must be positive. For the horizontal motion we apply Newton's second law,

$$F = Ma \tag{1.21}$$

The frictional force F is bounded by

$$F \le \mu N_2 \tag{1.22}$$

The maximum friction force occurs just as the racer tires begin to slip relative to the drag strip, because the coefficient of kinetic friction is smaller than the coefficient of static friction. For maximal initial acceleration we must have the maximum friction force $F = \mu N_2$. Referring back to (1.20), a maximal $N_2 = Mg$ is obtained when $N_1 = 0$, that is, when the back wheels completely support the racer. The greatest possible acceleration is then

$$a_{\text{max}} = \frac{\mu (N_2)_{\text{max}}}{M} = \mu g \tag{1.23}$$

We see that the optimum acceleration is independent of the racer's mass. Under normal conditions the coefficient of friction μ between rubber and concrete is about unity. Thus a racer can achieve an acceleration of about 9.8 m/s^2. In actual design a small normal force N_1 on the front wheels is allowed for steering purposes.

The standard drag strip is ≈ 400 m (1/4 mi) in length. If we assume that the racer can maintain the maximum acceleration for the duration of a race and that the coefficient of friction is constant, we can calculate the final velocity and the elapsed time. The differential form of the second law is

$$F = Ma = M\frac{dv}{dt} = M\ddot{x} \tag{1.24}$$

When the acceleration a is constant, a single integration

$$\int_{v_0}^{v} dv = a \int_0^t dt \tag{1.25}$$

gives

$$v - v_0 = at \tag{1.26}$$

Using $dx = vdt$, a second integration

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + at)dt \tag{1.27}$$

yields

$$x - x_0 = v_0 t + \tfrac{1}{2}at^2 \tag{1.28}$$

We can eliminate t from (1.26) and (1.28) to obtain

$$v^2 = v_0^2 + 2a(x - x_0) \tag{1.29}$$

Substituting $a = 9.8\,\text{m/s}^2$, $x = 0.40\,\text{km}$, $x_0 = 0$ and $v_0 = 0$, we find $v = 89\,\text{m/s}$ (or $320.4\,\text{km/h}$)! The time elapsed, $t = v/a$, is about 9 s. For comparison, the world drag-racing records (with a piston engine) as of 1992 are $v = 134.8\,\text{m/s}$ ($485.3\,\text{km/h}$) for velocity and 4.80 s for elapsed time. (These records were set in different races.) With tires that are several times wider than automobile tires and have treated surfaces, coefficients of friction considerably greater than $\mu = 1$ are realized in drag racing. The rubber laid down by previous racers in effect gives a rubber-rubber contact which also increases the coefficient of friction. Aerodynamic effects are important as well. The drag force from wind resistance reduces the speed of a racer, while a negative lift force on the back wheels can be produced by wind resistance against an up-tilted rear wing found on many racers, which increases the normal force, giving greater traction and allowing larger acceleration.

1.4 Sport Parachuting: Aerodynamic Drag

The sport of skydiving visually illustrates the effect of the viscous frictional force of (1.16). Immediately upon leaving the aircraft, the jumper accelerates downward due to the gravity force. As his velocity increases, the air resistance exerts a greater retarding force, and eventually approximately balances the pull of gravity. From this time onward the descent of the diver is at a uniform rate, called the *terminal velocity*. The terminal velocity in a spread-eagle position is roughly 120 mi/h. By assuming a vertical head-down position, the diver can decrease his cross sectional

area (perpendicular to the direction of motion) thereby lowering the air resistance [smaller value of c in (1.16)], and increase his terminal velocity of descent. Eventually, of course, the diver opens his parachute. This dramatically increases the air resistance and correspondingly reduces his terminal velocity, to allow a soft impact with the ground.

To analyze the physics of skydiving, we shall assume that the motion is vertically downward and choose a coordinate system with $x = 0$ at the earth's surface and positive upward. In this coordinate frame, downward forces are negative. We approximate the external force on the diver as

$$F = -mg + cv^2 \tag{1.30}$$

The frictional force is positive, as required for an upward force. The terminal velocity is reached when the opposing gravity and frictional forces balance, giving $F = 0$. Under this condition, the terminal velocity is

$$v_t = \sqrt{\frac{mg}{c}} \tag{1.31}$$

To solve the differential equation of motion,

$$F = m\frac{dv}{dt} = -mg + cv^2 \tag{1.32}$$

we rearrange the factors and integrate

$$\int_0^v \frac{dv}{v_t^2 - v^2} = -\frac{g}{v_t^2}\int_0^t dt \tag{1.33}$$

In (1.32) the frictional coefficient c has been replaced by v_t from (1.31). We obtain

$$\frac{1}{2v_t}\ln\left(\frac{v_t + v}{v_t - v}\right) = -\frac{g}{v_t^2}t \tag{1.34}$$

which can be inverted to express v in terms of t,

$$v = -v_t\frac{1 - \exp(-2gt/v_t)}{1 + \exp(-2gt/v_t)} \tag{1.35}$$

At large times the decreasing exponentials go to zero rapidly and v approaches the terminal velocity,

$$v \to -v_t \tag{1.36}$$

Although the limiting velocity is exactly reached only at infinite time, it is approximately reached for times $t \gg v_t/2g$. A typical value for v_t

on a warm summer day is 54 m/s (194.4 km/h) for a 70 kg diver in a spread-eagle position. After a time

$$t = \frac{2v_t}{g} = \frac{2(54)}{9.8} = 11\,\mathrm{s} \tag{1.37}$$

the sky diver would be traveling about 52 m/s with his parachute unopened! The velocity of the diver as a function of time is plotted in Figs. 1-2 and 1-3. To calculate the distance the diver has fallen after a specific elapsed time, we integrate $dx = v\,dt$ using (1.35),

$$\int_h^x dx = -v_t \int_0^t \left(1 - \frac{2\exp(-2gt/v_t)}{1+\exp(-2gt/v_t)}\right) dt \tag{1.38}$$

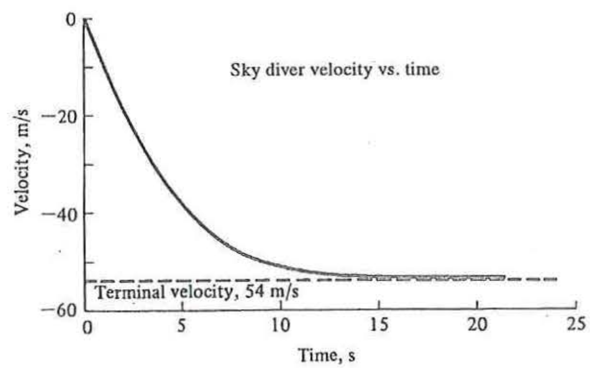


FIGURE 1-2. Velocity of a sky diver as a function of time for a terminal velocity of 54 m/s.

The result of the integration is

$$h - x = v_t\left[t - \frac{v_t}{g}\ln\left(\frac{2}{1+\exp(-2gt/v_t)}\right)\right] \tag{1.39}$$

At time $t = 2v_t/g$, the diver has fallen a distance $(h - x)$, given by

$$h - x = \frac{{v_t}^2}{g}\left[2 - \ln\left(\frac{2}{1+e^{-4}}\right)\right] = \frac{(54)^2}{9.8}(2 - 0.7) = 385\,\mathrm{m} \tag{1.40}$$

Sky divers normally free-fall about 1,400 m (in 30 s) so much of the descent is at terminal velocity.

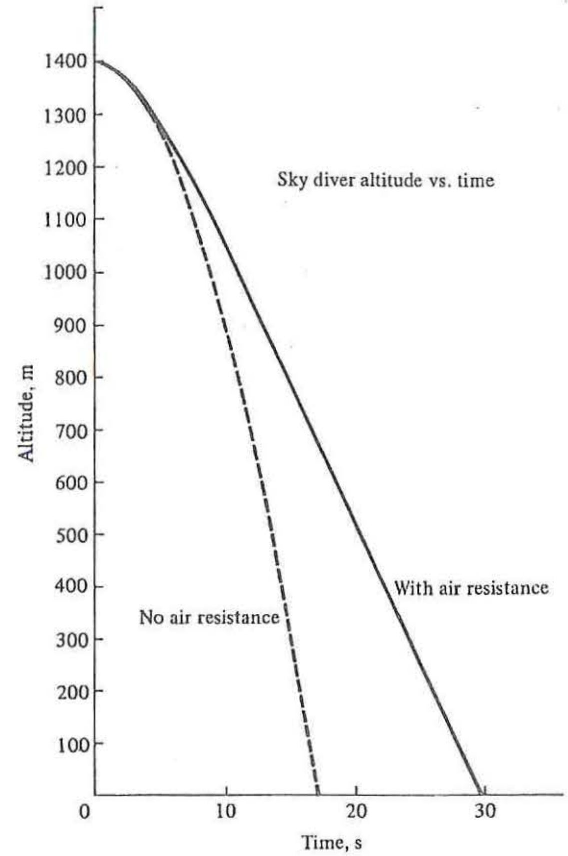


FIGURE 1-3. Altitude of a sky diver with unopened parachute as a function of time (for a terminal velocity of 54 m/s).

Finally, let us use the drag coefficient formula of (1.17) to estimate the free fall terminal velocity of a sky diver. By (1.31) and (1.17) we have

$$v_t = \sqrt{\frac{mg}{\frac{1}{2}C_D S\rho}} \tag{1.41}$$

Assuming that in a spread-eagle position $C_D \simeq 0.5$ and $S \simeq 1\ \mathrm{m}^2$ and that the air density is $1\,\mathrm{kg/m^3}$ we find for a 70 kg diver,

$$v_t = \sqrt{\frac{70(9.8)}{0.25(1)(1)}} = 52\ \mathrm{m/s} \tag{1.42}$$

very near the actual value. The excellent agreement is fortuitous but the ability to make such estimates of the drag force is certainly useful.

1.5 Archery: Spring Force

The force exerted on an arrow by an archer's bow can be approximated by the spring force of (1.11). A 134 Newton bow with a 0.72 meter draw d has a spring constant k given by

$$k = \frac{|F|}{d} = \frac{134}{0.72} = 186\,\mathrm{kg/s^2} \tag{1.43}$$

After release of the bowstring, the motion of the arrow of mass m is described by the second law,

$$m\frac{dv}{dt} = -kx\,, \qquad x < 0 \tag{1.44}$$

until it leaves the bowstring at $x = 0$. Here we neglect the mass of the bowstring. To integrate this differential equation for the velocity, we use the chain rule of differentiation

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v \tag{1.45}$$

Substituting into (1.44), rearranging factors, and integrating we obtain

$$m\int_0^v v\,dv = -k\int_{-d}^0 x\,dx \tag{1.46}$$

or

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kd^2 \tag{1.47}$$

Thus the velocity of the arrow as it leaves the bowstring is given by

$$v = d\sqrt{\frac{k}{m}} \tag{1.48}$$

The longer the draw and the stronger the bow, the higher the arrow velocity. For a typical target arrow, with weight $m = 23$ g, the velocity is

$$v = 0.72\sqrt{\frac{186}{23\times 10^{-3}}} = 65\,\mathrm{m/s} \tag{1.49}$$

This is almost double the maximum speed of a fastball thrown by a professional baseball player!

1.6 Methods of Solution

For the general motion of a particle in one dimension, the equation of motion is

$$m\ddot{x} = F(x, \dot{x}, t) \tag{1.50}$$

Since this is a second-order differential equation, the solution for x as a function of t involves two arbitrary constants. These constants can be fixed from physical conditions, such as the position and velocity at the initial time. In the examples of § 1.3 to 1.5, we have introduced several techniques for solving (1.50). In the case where F depends on only one of the variables x, $\dot{x}$, or t, the formal solution of (1.50) is straightforward. We now run through the methods of solution to the differential equations of motion for these specific classes of force laws.

For a force that depends only on x, we may use the chain rule of (1.45), and integrate (1.50) to obtain

$$m\int^v v'dv' = \int^x F(x')\,dx' + C_1 \tag{1.51}$$

where C_1 is a constant of integration. Here we have used primes to denote the dummy variables of integration. The resulting expression for $v(x)$ is

$$v = \sqrt{\frac{2}{m}}\sqrt{\int^x F(x')\,dx' + C_1} \tag{1.52}$$

This method of solution was employed in the archery discussion of § 1.5. The solution for $x(t)$ is found by substituting $v = \dot{x}$ in (1.52), rearranging factors so as to separate the variables, and integrating, to get

$$\int^x \frac{dx'}{\sqrt{\int^{x'} F(x'')dx'' + C_1}} = \sqrt{\frac{2}{m}}\int^t dt' + C_2 \tag{1.53}$$

The integration constants C_1 and C_2 can be fixed from the initial velocity and position.

With a velocity-dependent force we can integrate (1.50) as follows:

$$m\int^v \frac{dv'}{F(v')} = \int^t dt' + C_1 \tag{1.54}$$

We used this technique in the sky-diving analysis of § 1.4. The result of the integration gives $v(t)$, which can then be integrated over t to find $x(t)$.

The solution of (1.50) for a time-dependent force $F(t)$ can be obtained from direct integration,

$$m\int^{v} dv' = \int^{t} F(t')\,dt' + C_1 \tag{1.55}$$

A second integration leads to the solution for $x(t)$,

$$m\int^{x} dx' = \int^{t}\left[\int^{t'} F(t'')dt'' + C_1\right] dt' + C_2 \tag{1.56}$$

If the force law depends on more than one variable, the techniques for finding analytical solutions, when they exist, are more complicated.

For the forces involved in many physical problems, (1.50) cannot be solved in closed analytical form. However, we can then resort to numerical methods which can be evaluated using computers. To illustrate the numerical approach, we assume that the position x_0 and velocity v_0 are known at the initial time t_0. The acceleration a_0 then is given by (1.50) as

$$a_0 = \frac{F(x_0, v_0, t_0)}{m} \tag{1.57}$$

After a short time interval Δt,

$$\begin{aligned} t_1 &= t_0 + \Delta t \\ x_1 &= x_0 + v_0\Delta t \\ v_1 &= v_0 + a_0\Delta t \end{aligned} \tag{1.58}$$

where we have neglected the change in a and v over Δt. This approximation becomes more accurate as the time increment Δt is made smaller. From these new values of the variables, we can calculate the new acceleration using (1.50)

$$a_1 = \frac{F(x_1, v_1, t_1)}{m} \tag{1.59}$$

By repetition of this procedure n times, we can calculate x and v at time $t_n = t_0 + n\,\Delta t$

$$\begin{aligned} x_n &= x_{n-1} + v_{n-1}\ \Delta t \\ v_n &= v_{n-1} + a_{n-1}\ \Delta t \end{aligned} \tag{1.60}$$

We thereby obtain a complete numerical solution to the equation of motion. The solution becomes more accurate as the time increment Δt is

made smaller. This illustrates that a unique solution to the differential equation of motion is always possible for any reasonable force law. For the numerical solution to a specific problem the use of more sophisticated numerical methods is usually desirable in order to increase the accuracy of the result for a given Δt.

1.7 Simple Harmonic Oscillator

Many common physical applications of the spring force involve oscillatory motion, such as vibrations of a mass attached to a spring. A system undergoing periodic steady-state motion under the action of a spring is called a *harmonic oscillator*. The motion is called *simple harmonic* when the restoring force is proportional to the displacement from an equilibrium position (for instance, proportional to the extension or compression of a spring). Any system in which there is a linear restoring force (such as AC circuits and certain servomechanisms) exhibits simple harmonic oscillations.

The equation of motion for a simple harmonic oscillator,

$$m\ddot{x} = -kx \tag{1.61}$$

with $k > 0$, can be solved by (1.52) and (1.53). However, we can cleverly construct the solution as follows. The functions $\cos\omega_0 t$ and $\sin\omega_0 t$ satisfy (1.61) if the angular frequency ω_0 is given by

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{1.62}$$

The general solution to (1.61) is a linear superposition of $\cos\omega_0 t$ and $\sin\omega_0 t$ solutions

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{1.63}$$

where A and B are arbitrary constants. An equivalent form of the solution is

$$x(t) = a\cos(\omega_0 t + \alpha) \tag{1.64}$$

with constants related by

$$A = a\cos\alpha \qquad B = -a\sin\alpha \tag{1.65}$$

The constant a is called the *amplitude* of the motion, and α is called the *initial phase*. The initial conditions can be used to specify the arbitrary constants a and α. From (1.64) the velocity of the oscillator is

$$v(t) = -a\omega_0 \sin(\omega_0 t + \alpha) \tag{1.66}$$

The period τ of the motion is the time required for the system to undergo a complete oscillation and return to the initial values of x and v. The period for the oscillator is

$$\tau = \frac{2\pi}{\omega_0} \tag{1.67}$$

The frequency of the oscillator (number of oscillations per unit time) is

$$\nu = \frac{1}{\tau} = \frac{\omega_0}{2\pi} \tag{1.68}$$

We can illustrate our harmonic-oscillator solution with the bow-and-arrow example of § 1.5. At $t = 0$ the bow is at full draw, $x = -d$, and the arrow velocity is zero. From (1.66) we find

$$\alpha = 0 \tag{1.69}$$

and from (1.64) we obtain

$$a = -d \tag{1.70}$$

The solution with proper boundary conditions is

$$x(t) = -d\cos\omega_0 t \tag{1.71}$$
$$v(t) = d\omega_0 \sin\omega_0 t \tag{1.72}$$

with $\omega_0 = \sqrt{k/m}$. As time increases from $t = 0$, x increases to zero at

$$t = \frac{1}{2}\left(\frac{\pi}{\omega_0}\right) \tag{1.73}$$

At this instant the arrow leaves the bowstring with velocity

$$v = d\omega_0 = d\sqrt{\frac{k}{m}} \tag{1.74}$$

which agrees with (1.48). For the bow described in § 1.5 the arrow-propulsion time from (1.73) is

$$t = \frac{\pi}{2}\sqrt{\frac{m}{k}} = \frac{\pi}{2}\sqrt{\frac{23 \times 10^{-3}}{186}} \approx \frac{1}{60} s \tag{1.75}$$

In our archery example the simple-harmonic-force law does not apply beyond this time (one-fourth of the period τ), as illustrated in Figs. 1-4 and 1-5.

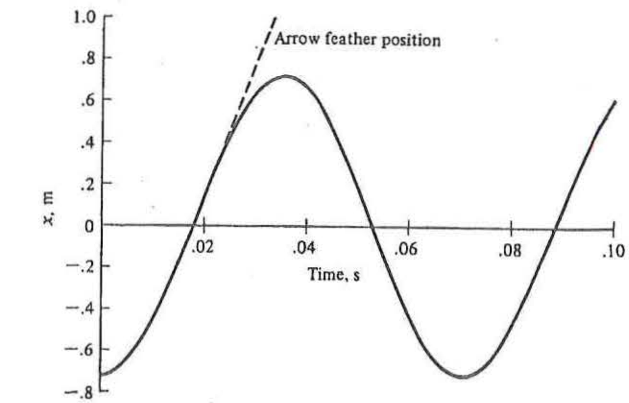


FIGURE 1-4. Displacement of a simple harmonic oscillator *vs.* time. The position of the feather end of the archer's arrow as a function of time is indicated by the dashed line after the arrow leaves the bow.

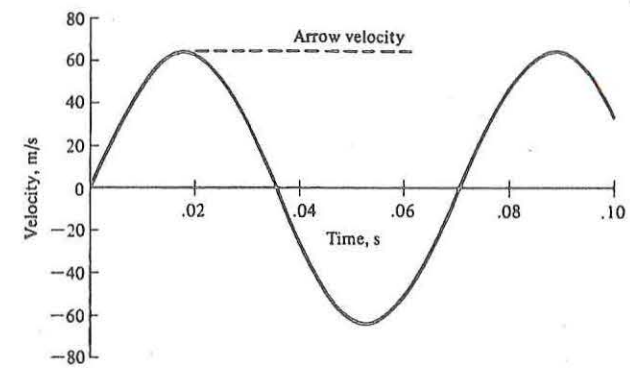


FIGURE 1-5. Velocity of a simple harmonic oscillator vs. time. The velocity of the arrow after it leaves the bow is indicated by the dashed line.

As another instructive example we consider the spring-mass system in Fig. 1-6. The spring, assumed massless, has a rest length ℓ. When the mass m is attached to the free end, the equation of motion in the absence of gravity is

$$m\ddot{x} = -k(x - \ell) \tag{1.76}$$

or

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \ell\,, \qquad \omega_0^2 = \frac{k}{m} \tag{1.77}$$

The solution is evidently of the form

$$x(t) = C + A\cos\omega_0 t + B\sin\omega_0 t \tag{1.78}$$

Substitution into (1.77) yields $C = \ell$ and we conclude that the motion consists of harmonic motion with angular frequency ω_0 about the equilibrium point $x = \ell$.

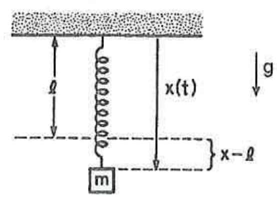


FIGURE 1-6. A mass m suspended by a spring of rest length ℓ undergoes vertical oscillations.

With gravity present, we must add mg to the forces acting on m and the equation of motion becomes

$$m\ddot{x} = -k(x - \ell) + mg \tag{1.79}$$

or

$$\ddot{x} + \omega_0^2 x = \omega_0^2 \ell + g \tag{1.80}$$

Comparing to the gravity-free case shows the equation of motion differs only by the constant on the right side. The solution to (1.80) is then (1.78) but with $C = \ell + \frac{mg}{k}$. The motion is again harmonic with angular frequency ω_0 except that the equilibrium point is $\ell + \frac{mg}{k}$. When the mass is at $\ell + \frac{mg}{k}$, the upward force due to the spring is $k(\frac{mg}{k}) = mg$, which just equals the weight force. The mass will remain at this position if released at rest there.

We conclude this section by solving the simple harmonic equation of motion (1.61) in a more systematic way. The equation of motion

$$\ddot{x} + \omega_0^2 x = 0\,, \qquad \omega_0 = \sqrt{\frac{k}{m}} \tag{1.81}$$

is a linear differential equation with constant coefficients; such an equation always has a solution of the form

$$x(t) = e^{\lambda t} \tag{1.82}$$

With this substitution, (1.81) becomes

$$(\lambda^2 + \omega_0^2)e^{\lambda t} = 0 \tag{1.83}$$

which requires that $\lambda^2 = -\omega_0^2$. Thus (1.82) is a solution if $\lambda = i\omega_0$ or $\lambda = -i\omega_0$ and so the linear superposition

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t} \tag{1.84}$$

is a solution; here c_1 and c_2 are constants (generally complex). Since by appropriately choosing these constants we can fit any initial conditions x_0 and $\dot{x}_0$, (1.84) is the general solution. Using the identity $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$ we can rewrite this general solution as

$$\begin{aligned} x(t) &= c_1(\cos\omega_0 t + i\sin\omega_0 t) + c_2(\cos\omega_0 t - i\sin\omega_0 t) \\ &= (c_1 + c_2)\cos\omega_0 t + i(c_1 - c_2)\sin\omega_0 t \end{aligned} \tag{1.85}$$

Since any physical quantity such as $x(t)$ must be real (no imaginary part) we must choose $c_2 = c_1^*$, where c_1^* is the complex conjugate of c_1. With $2\mathcal{R}e\, c_1 = A$ and $2\mathcal{I}m\, c_1 = -B$, we obtain the result in (1.63).

1.8 Damped Harmonic Motion

In almost all physical problems frictional forces play a role. For example, a harmonic oscillator that is subject to a damping force has an amplitude that decreases with time. For this reason, and also because a damped harmonic oscillator applies to such a broad range of physical phenomena, we treat its solution at some length. The form (1.14) chosen for the frictional force is linear in the velocity; the equation of motion is then linear in both x and its time derivatives and is solvable analytically.

The equation of motion of the damped harmonic oscillator is

$$m\ddot{x} = -kx - b\dot{x} \tag{1.86}$$

We define the damping constant $\gamma = \frac{1}{2}(b/m)$ and the natural frequency $\omega_0 = \sqrt{k/m}$ to express (1.86) in the form

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{1.87}$$

Like the undamped harmonic oscillator equation of motion, (1.81), this is a linear differential equation with constant coefficients so again

$$x(t) = e^{\lambda t} \tag{1.88}$$

is a solution. Substituting this into (1.87) we find

$$(\lambda^2 + 2\gamma\lambda + \omega_0^2)e^{\lambda t} = 0 \tag{1.89}$$

which is satisfied only if the term in parentheses vanishes. Solving the quadratic equation, the possible values of λ are

$$\lambda = -\gamma \pm \sqrt{\gamma^2 - {\omega_0}^2} \equiv -\gamma \pm \Omega \tag{1.90}$$

where we have defined

$$\Omega \equiv \sqrt{\gamma^2 - \omega_0^2} \tag{1.91}$$

The qualitative nature of the solution depends on the relative magnitude of the frictional coefficient γ and the natural frequency ω_0. We distinguish the three cases:

$$\begin{array}{lll} \text{I.} & \gamma > \omega_0\,, & \Omega \text{ real} \\ \text{II.} & \gamma = \omega_0\,, & \Omega \text{ zero} \\ \text{III.} & \gamma < \omega_0\,, & \Omega \text{ imaginary} \end{array} \tag{1.92}$$

For $\Omega \neq 0$ the general solution is a superposition of $e^{\lambda t}$ terms with both possible values of λ. In case I the solution is

$$\text{I.} \qquad \begin{aligned} x(t) &= c_1 e^{-(\gamma-\Omega)t} + c_2 e^{-(\gamma+\Omega)t} \\ &= e^{-\gamma t}(c_1 e^{\Omega t} + c_2 e^{-\Omega t}) \end{aligned} \tag{1.93}$$

If $\Omega = 0$ the two terms in (1.93) have the same t-dependence. Then, since the expression depends only on the one constant $c_1 + c_2$, (1.93) is not

the general solution of the second order differential equation. Treating the $\Omega = 0$ case as a limit $\Omega \to 0$ we can expand the exponentials

$$e^{\pm\Omega t} = 1 \pm \Omega t + \mathcal{O}(\Omega^2) \tag{1.94}$$

and group the terms in (1.93) as

$$x(t) = e^{-\gamma t}\Big[(c_1 + c_2) + (c_1 - c_2)\Omega t\Big] + \mathcal{O}(\Omega^2) \tag{1.95}$$

Then defining $C = c_1 + c_2$ and $D = (c_1 - c_2)\Omega$, the solution for $\Omega = 0$ is

$$\text{II.} \qquad x(t) = e^{-\gamma t}[C + Dt] \tag{1.96}$$

In case III, we express

$$\Omega = \sqrt{\gamma^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \gamma^2} = i\Omega' \tag{1.97}$$

in terms of the real quantity

$$\Omega' = \sqrt{\omega_0^2 - \gamma^2} \tag{1.98}$$

Then the form of the solution (1.93) is

$$\text{III.} \qquad x(t) = e^{-\gamma t}\left(c_1 e^{i\Omega' t} + c_2 e^{-i\Omega' t}\right) \tag{1.99}$$

For $x(t)$ to be real the constants c_1 and c_2 must be complex conjugates, $c_2 = c_1^*$. Thus the solution can be expressed in the form

$$x(t) = 2e^{-\gamma t}\mathcal{R}e\left(c_1 e^{i\Omega' t}\right) \tag{1.100}$$

Writing c_1 in polar form as $c_1 = \frac{1}{2}ae^{i\alpha}$ where a and α are real, we obtain

$$\text{III.} \qquad x(t) = ae^{-\gamma t}\cos(\Omega' t + \alpha) \tag{1.101}$$

The two constants which appear in the above solutions can be related to the initial conditions $x(0) \equiv x_0$ and $\dot{x}(0) \equiv v_0$ at time $t = 0$. After solving for the constants from the initial conditions, the solutions are of the forms

$$\text{I. } x(t) = \frac{1}{2}\left[x_0 + \frac{(v_0 + \gamma x_0)}{\Omega}\right] e^{-(\gamma-\Omega)t} + \frac{1}{2}\left[x_0 - \frac{(v_0 + \gamma x_0)}{\Omega}\right] e^{-(\gamma+\Omega)t} \tag{1.102}$$

$$\text{II.} \qquad x(t) = e^{-\gamma t}\left[x_0 + (v_0 + \gamma x_0)\, t\right] \tag{1.103}$$

$$\text{III.} \qquad x(t) = a e^{-\gamma t} \cos(\Omega' t + \alpha) \tag{1.104}$$

with $a = \left(\omega_0^2 x_0^2 + 2\gamma v_0 x_0 + v_0^2\right)^{1/2}/\Omega'$ and $\tan\alpha = -(v_0 + \gamma x_0)/x_0\Omega'$.

In all three cases the amplitude of the displacement decays exponentially with time, although in II the exponential factor is multiplied by a linear function of t. At large times the rates of falloff are characterized by the exponentials:

$$\begin{array}{lll} \text{I.} & e^{-(\gamma-\Omega)t} & \gamma > \omega_0 \text{ (overdamped)} \\ \text{II.} & e^{-\gamma t} * \text{(linear function of } t) & \gamma = \omega_0 \text{ (critically damped)} \\ \text{III.} & e^{-\gamma t} * \text{(sinusoidal function of } t) & \gamma < \omega_0 \text{ (underdamped)} \end{array} \tag{1.105}$$

Illustrations of the time dependences for the three cases are given in Fig. 1-7 for the initial conditions $x = x_0$, $v_0 = 0$. An exception to the above rates of decrease occurs when the initial conditions are such that the coefficient of the $e^{-(\gamma-\Omega)t}$ term of solution I vanishes. In that circumstance, the mass returns to rest like $e^{-(\gamma+\Omega)t}$.

There are endless applications of damped harmonic oscillators. The pneumatic spring return on a door represents an everyday situation where solution II is the ideal. Upon releasing the door with no initial velocity, we want it to close as rapidly as possible without slamming. Equations (1.105) indicate that solution II should be selected; the spring-tube system should be designed with $\gamma = \omega_0$. Solution III might close the door faster, due to the vanishing of the cosine factor in (1.104), but this would let the door slam! On the other hand, solution III describes physical systems that undergo damped periodic oscillations.

The behavior of simple electric circuits is determined by a differential equation which has the same mathematical form as the damped harmonic

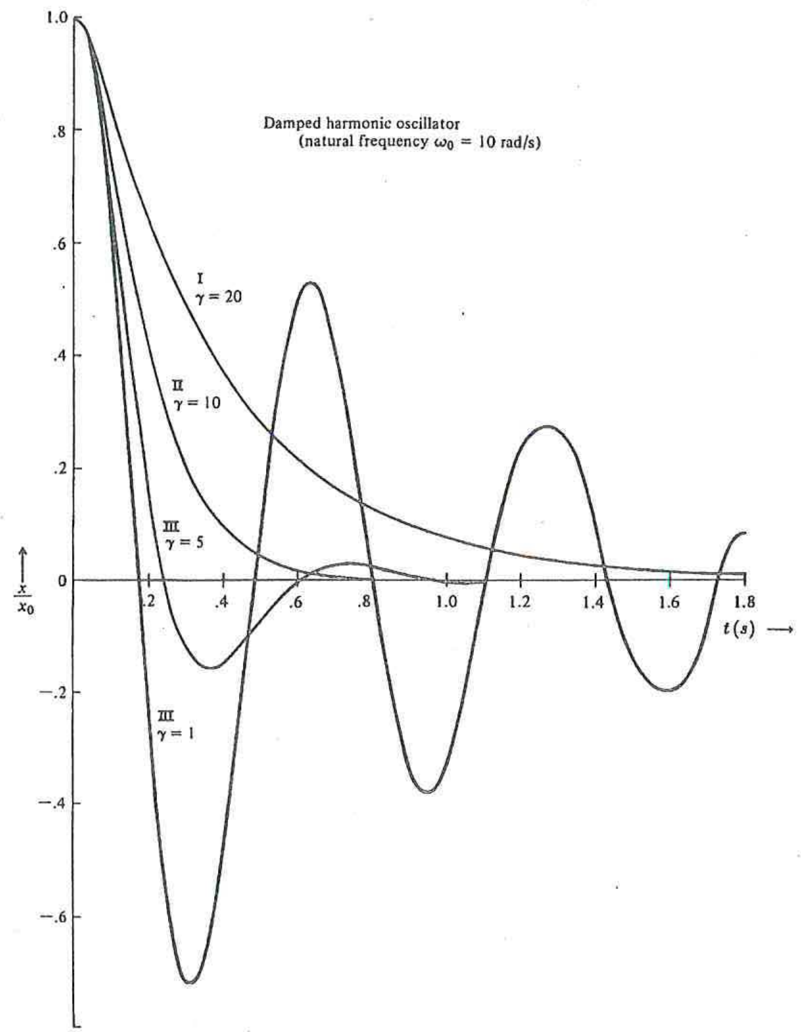


FIGURE 1-7. Time dependence of the displacement of a damped harmonic oscillator for the initial conditions $x = x_0$, $v = 0$. The natural frequency of the oscillator is $\omega_0 = 10$ rad/s. Results for various strengths of the damping constant γ are illustrated.

oscillator. As an example we consider the circuit of Fig. 1-8 with an inductor L, resistor R, and capacitor C in series. When the switch is closed, the sum of the voltage drops across the elements of the circuit must add up to zero. This leads to the differential equation

$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0 \tag{1.106}$$

where $i(t)$ is the current flowing in the circuit and $q(t)$ is the charge on one of the capacitor plates. Since $i = dq/dt = \dot{q}$, the circuit equation can be written as

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0 \tag{1.107}$$

This equation has the form of the damped-harmonic-oscillator equation (1.86) with the following correspondences:

$$\begin{aligned} x &\to q & \gamma &= \frac{b}{2m} \to \frac{R}{2L} \\ m &\to L & \omega_0 &= \sqrt{\frac{k}{m}} \to \sqrt{\frac{1}{LC}} \\ b &\to R & \Omega &= \sqrt{\gamma^2 - \omega_0^2} \to \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \\ k &\to \frac{1}{C} & \Omega' &= \sqrt{\omega_0^2 - \gamma^2} \to \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \end{aligned} \tag{1.108}$$

Since it is often far easier to connect circuit elements than to build and test a mechanical system, this analogy has been of considerable practical importance.

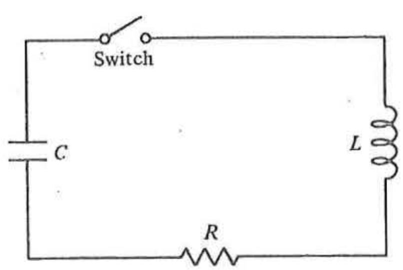


FIGURE 1-8. Simple L, R, C electric series circuit.

If the circuit in Fig. 1-8 is in a static state, when the switch is closed at time $t = 0$ the initial conditions are

$$\begin{aligned} q(t=0) &= q_0 = CV_0 \\ i(0) &= \dot{q}(t=0) = 0 \end{aligned} \tag{1.109}$$

where V_0 is the voltage across the capacitor. By reference to (1.102)–(1.104) the solutions for the charge as a function of time are

I.
$$\begin{aligned} \frac{R}{2L} &> \sqrt{\frac{1}{LC}} \qquad (\gamma > \omega_0\,,\ \text{overdamped}) \\ q(t) &= q_0\left[\left(1+\frac{\gamma}{\Omega}\right)e^{-(\gamma-\Omega)t} + \left(1-\frac{\gamma}{\Omega}\right)e^{-(\gamma+\Omega)t}\right] \end{aligned} \tag{1.110}$$

II.
$$\begin{aligned} \frac{R}{2L} &= \sqrt{\frac{1}{LC}} \qquad (\gamma = \omega_0\,,\ \text{critically damped}) \\ q(t) &= q_0(1+\gamma t)e^{-\gamma t} \end{aligned} \tag{1.111}$$

III.
$$\begin{aligned} \frac{R}{2L} &< \sqrt{\frac{1}{LC}} \qquad (\gamma < \omega_0\,,\ \text{underdamped}) \\ q(t) &= (\Omega')^{-1}\omega_0 q_0 e^{-\gamma t}\cos\left(\Omega' t - \arctan\frac{\gamma}{\Omega'}\right) \end{aligned} \tag{1.112}$$

For a circuit with a voltage source, as in Fig. 1-9, the sum of the voltage drops around the circle must equal 0. Thus the differential equation for the circuit in Fig. 1-9 is

$$L\frac{di}{dt} + Ri + \frac{q}{C} = V(t) \tag{1.113}$$

where $V(t)$ is the voltage of the generator. This differential equation is of the form of the equation of motion for a damped harmonic oscillator subjected to an external force, a topic which we take up in the following section.

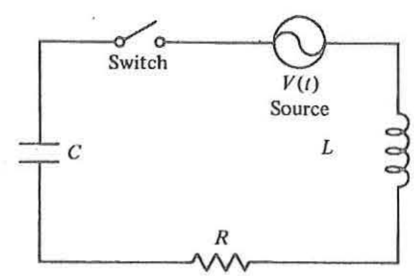


FIGURE 1-9. Series L, R, C circuit with a voltage generator $V(t)$.

1.9 Damped Oscillator With Driving Force: Resonance

Numerous physical systems can be described in terms of a damped harmonic oscillator driven by an external force that oscillates sinusoidally with time as

$$F(t) = F_0 \cos\omega t = mf \cos\omega t \tag{1.114}$$

where we have introduced $f = F_0/m$ for later convenience. The equation of motion in (1.87) gets modified to

$$\ddot{x} + 2\gamma\dot{x} + {\omega_0}^2 x = f\cos\omega t \tag{1.115}$$

A *particular solution* to this inhomogeneous linear differential equation is most readily obtained by using complex numbers and solving for a related equation with a complex driving force. For this purpose we introduce

$$\begin{aligned} z &= x + iy \\ e^{i\omega t} &= \cos\omega t + i\sin\omega t \end{aligned} \tag{1.116}$$

and observe that the real part of

$$\ddot{z} + 2\gamma\dot{z} + {\omega_0}^2 z = fe^{i\omega t} \tag{1.117}$$

is identical with (1.115). This latter form is more convenient to solve. Once we find the solution for z, the physical displacement x is obtained from $x = \mathcal{R}e\, z$. Note that if the left-hand side of (1.117) were not linear in z this method would not work. Since the first and second derivatives of $e^{i\omega t}$ are $i\omega e^{i\omega t}$ and $-\omega^2 e^{i\omega t}$, there is a solution with the time dependence $e^{i\omega t}$. Thus, as a possible solution to (1.117), we try

$$z = \frac{f}{R}e^{i\omega t} \tag{1.118}$$

where $1/R$ is a time-independent response factor. The differential equation is satisfied by this choice if

$$\left[(i\omega)^2 + 2\gamma(i\omega) + \omega_0^2\right]\frac{f}{R} = f \tag{1.119}$$

or

$$R = \omega_0^2 - \omega^2 + 2i\gamma\omega \tag{1.120}$$

The complex factor R can be written in polar form

$$R = re^{i\theta} \tag{1.121}$$

where

$$r^2 = |R|^2 = \left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2 \tag{1.122}$$

and

$$\tan\theta = \left(\frac{\mathcal{I}mR}{\mathcal{R}eR}\right) = \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \tag{1.123}$$

The angle θ lies between 0 and π. Using (1.116), (1.118), and (1.121), we arrive at the desired solution to (1.115)

$$x = \mathcal{R}ez = \mathcal{R}e\left(\frac{f}{r}e^{i(\omega t - \theta)}\right) \tag{1.124}$$

or

$$x(t) = \frac{f}{r}\cos(\omega t - \theta) \tag{1.125}$$

The response $x(t)$ to the force $mf\cos\omega t$ is thus proportional to $1/r$. The response oscillates with a phase $(\omega t - \theta)$ that lags the oscillations of the force by a phase angle θ.

Both r and θ depend on the relative size of the driving frequency ω and natural frequency ω_0. For small damping $\gamma \ll \omega_0$ and values of ω near to ω_0, we can make the following approximations in (1.121) and (1.122):

$$r^2 = (\omega_0 - \omega)^2(\omega_0 + \omega)^2 + 4\gamma^2\omega^2 \approx 4\omega_0^2\left[(\omega_0 - \omega)^2 + \gamma^2\right] \tag{1.126}$$

$$\tan\theta = \frac{2\gamma\omega}{(\omega_0 - \omega)(\omega_0 + \omega)} \approx \frac{\gamma}{\omega_0 - \omega} \tag{1.127}$$

From these approximate expressions, we see that r^2 has a minimum when the driving force is at the natural frequency of the oscillator, $\omega = \omega_0$. The large response $x(t)$ produced by a driving frequency in the vicinity $\omega = \omega_0$ is called a *resonance*. The magnitude r_m of r at the resonance frequency $\omega = \omega_0$ is governed by the size of the frictional coefficient γ.

$$r_m = 2\omega_0\gamma \tag{1.128}$$

The width of the resonance is defined as the difference of the two values of ω at which r^2 is twice its minimum value. From (1.126) and (1.128) these values are $\omega = \omega_0 \pm \gamma$ and thus the width is 2γ. The resonance becomes narrower and the maximum displacement x larger as friction is made smaller. Plots of $[r(\omega_0)/r(\omega)]^2$ and $\theta(\omega)$ are shown in Figs. 1-10 and 1-11. The phase lag θ is 90° at resonance. At small frequencies ω, the phase lag tends to zero, and far above resonance it approaches 180°, as is evident from (1.123) or (1.120). Resonance phenomena analogous to that discussed here play an extremely important role in all branches of physics and engineering.

The solution we have been discussing is known as a *particular solution* since there are no integration constants. This particular solution is often called the *steady state* solution. It is not the most general solution since it does not match a general initial state of the oscillator. The general solution to the forced-oscillator differential equation is obtained by adding to the particular solution in (1.125) the general solution of the homogeneous equation (*i.e.*, the oscillation equation with no driving force). The result for the underdamped case is

$$x(t) = ae^{-\gamma t}\cos(\Omega' t + \alpha) + \frac{f}{\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{\frac{1}{2}}}\cos\left(\omega t - \arctan\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \qquad (1.129)$$

The sum satisfies (1.117) and contains two arbitrary constants, a and α. The initial conditions determine these constants. The term with the decaying exponential is called a transient—it vanishes at large times. The force-dependent term describes the steady-state oscillatory motion of the harmonic system.

Any periodic force can be Fourier-analyzed into an infinite series of $\cos(n\omega t + \phi_n)$ terms

$$F(t) = m\sum_n f_n \cos(n\omega t + \phi_n) \qquad (1.130)$$

where F_n and ϕ_n are constants and the period is $2\pi/\omega$. The solution (1.129) for a driving force $F_0 \cos\omega t$ can be used for a force $F_n\cos(n\omega t + \phi_n)$. Then the solution for a superposition of driving frequencies in (1.130) can be obtained as a summation over solutions with driving frequencies $n\omega$.

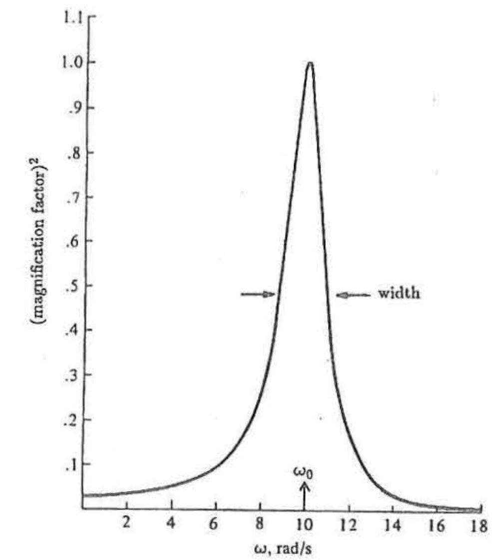


FIGURE 1-10. Square of the magnification factor, $\left[r(\omega = \omega_0)/r(\omega)\right]^2$, as a function of driving frequency ω for forced oscillator of natural frequency $\omega_0 = 10$ and damping constant $\gamma = 1$.

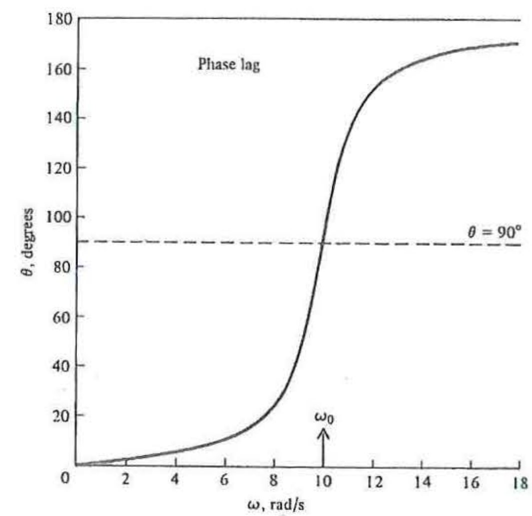


FIGURE 1-11. Phase lag θ as a function of driving frequency ω for the forced oscillator.

PROBLEMS

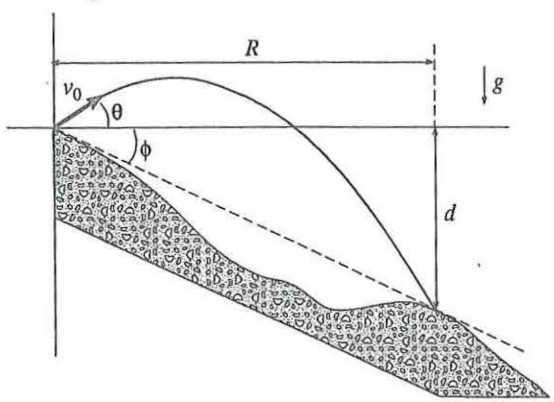
1.2 Interactions

1-1. An athlete can throw a javelin 60 m from a standing position. If he can run 100 m at constant velocity in 10 s, how far could he hope to throw the javelin while running? Neglect air resistance and the height of the thrower in the interest of simplicity. (*Hint: derive an expression for the distance R in terms of the initial angle θ to the horizontal and maximize R.*) Compare your answer with a world-class throw of 105 m for the javelin.

1-2. A world class shotputter can put a 7.26 kg shot a distance of 22 m. Assume that the shot is constantly accelerated over a distance of 2 m at an angle of 45 degrees and is released at a height of 2 m above the ground. Estimate the weight that this athlete can lift with one hand.

1-3. For the shotput of Problem 1-2 determine the initial angle θ of the trajectory to maximize the distance R of the put. Approximate the value of v_0 by that obtained in Problem 1-2. A photographic study found that expert athletes have learned by trial and error to release the shotput at this optimum angle.

1-4. A projectile is shot from the origin with initial velocity v_0 and inclination angle θ as shown.



Show the following:

a) The range R (maximum horizontal distance), v_0, θ and the drop d

are related by

$$R \sin 2\theta + d(1 + \cos 2\theta) = R^2/R_0$$

where

$$R_0 \equiv v_0^2/g$$

b) The condition for maximum range R_m is

$$\tan 2\theta_m = R_m/d$$

[Note that if $d = 0$ then $\theta_m = 45°$.]

c) If the land falls off with a constant slope angle ϕ (*i.e.*, $d = R_m \tan\phi$) then the maximum range angle θ_m and ϕ are related by

$$2\theta_m + \phi = 90°$$

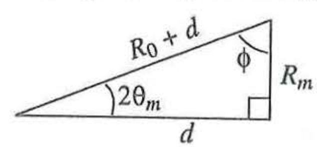
[Note that if $\phi = 0$ then $\theta_m = 45°$.]

d) The maximum range is given by

$$R_0^2 + 2dR_0 = R_m^2$$

[Note that if $d = 0$ then $R_m = R_0$.]

e) The optimum angle, maximum range, slope to impact angle ϕ, and the elevation drop d satisfy the triangle relation



1-5. A perfectly flexible cable has length l. Initially, the cable is at rest, with a length x_0 of it hanging vertically over the edge of a table. Neglecting friction, compute the length hanging over the edge after a time t. Assume that the sections of the cable remain straight during the motion.

1-6. A particle of mass m, initially at rest, moves on a horizontal line subject to a force $F(t) = ae^{-bt}$. Find its position and velocity as a function of time.

1.4 Drag Force

1-7. A boat is slowed by a drag force $F(v)$. Its velocity decreases according to the formula

$$v = c^2(t - t_1)^2$$

where c is a constant and t_1 is the time at which it stops. Find the force $F(v)$ as a function of v.

1-8. A mass m sliding horizontally is subject to a viscous drag force. For an initial velocity v_0 (at $x = t = 0$) and a retarding force $F = -b\dot{x}$ find the velocity as a function of distance, $v(x)$, and show that the mass moves a finite distance before coming to rest. For the same initial conditions and a retarding force $F = -c\dot{x}^2$ find $v(x)$ and $x(t)$, and show that the mass never comes to rest.

1-9. Integrate the equation of motion in (1.32) to directly find the velocity as a function of distance fallen for a sky diver in free fall. At what free-fall distance does the velocity reach two-thirds of the terminal velocity? Assume that $v_t = 54\,\text{m/s}$.

1-10. A diver of mass m begins a descent from a 10 meter diving board with zero initial velocity.

a) Calculate the velocity v_0 on impact with the water and the approximate elapsed time from dive until impact.

b) Set up the equation of motion for vertical descent of the diver through the water, assuming that the buoyancy force balances the gravity force underwater and the drag force is given by (1.16). Solve for the velocity v as a function of the depth x under water and impose the boundary condition $v = v_0$ at $x = 0$.

c) If the constant c in (1.16) is given by $c/m = 0.4$ (meter)$^{-1}$, estimate the depth at which $v = \frac{1}{10}v_0$.

d) Solve for the time under water in terms of the depth. How long does it take for the diver to reach the bottom of a 5 m deep pool?

1-11. A ball of mass m is thrown vertically upward with initial velocity v_i. If the air resistance is proportional to v^2 and the terminal velocity is v_t, show that the ball returns to its initial position with velocity v_f given by

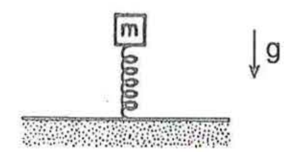
$$\frac{1}{v_f^2} = \frac{1}{v_i^2} + \frac{1}{v_t^2}$$

1-12. A bicyclist is able to pedal at a maximum speed v_0 on the level with no wind. If there is a wind force w parallel to the biker's path the biker will slow down or speed up. The air resistance is proportional to the square of the air speed. The biker's power output is equal to the applied force times the ground velocity. Assume the power output is constant and that there are no other power losses. Find an equation that relates v to v_0 and w. Compute the velocity numerically for $v_0 = 15$ m/s in the cases of a head wind $w = 5$ m/s and a tail wind $w = 5$ m/s.

1-13. A drag racer experiences a retarding force due to wind resistance that is proportional to the square of the racer's velocity. Assuming that the racer is designed for optimum acceleration, set up the equation of motion and derive a relation between v and t. Also derive a relation between v and x. Eliminate the coefficient of friction and solve the resulting equation numerically for the terminal velocity that can reproduce the 1988 world record of $v = 129.1$ m/s, $t = 4.99$ s for $x = 0.4$ km. Then determine the coefficient of friction.

1.5 Spring Force

1-14. A massless spring of rest length ℓ and spring constant k has a mass m attached to one end. The system is set on a table with the mass vertically above the spring as shown.



a) What is the new equilibrium height of the mass above the table?

b) The spring is compressed a distance c below the new equilibrium point and released. Find the motion of the mass assuming the free end of the spring remains in contact with the table.

c) Find the critical compression for which the spring will break contact with the table.

1-15. An archer using the equipment described in § 1.5 aims horizontally at a target 50 m away.

a) How far below the aiming point will the arrow strike? (Neglect air resistance.)

b) At what angle should the arrow be released so as to hit the target?

c) What would be the maximum possible flight distance on level ground? (Neglect air resistance and the height of the archer.)

d) Suppose that the arrow is released at a height of 1.6 m above the ground (typical shoulder-height of a person) at the angle found in part b) above. Calculate the horizontal distance at which the arrow would hit the ground.

1.7 Simple Harmonic Oscillator

1-16. Solve the damped unforced oscillator by the following method. Define a new variable y by

$$x = e^{\beta t} y$$

Substitute into the equation of motion (1.87) to find the equation satisfied by $y(t)$. Choose β such that the coefficient of $\dot{y}$ vanishes and solve in the underdamped, critical, and overdamped cases.

1-17. Show that the underdamped oscillator solution (1.104) can be expressed as $x(t) = x_0 e^{-\gamma t}\left[\cos \Omega' t + \left(\frac{v_0+\gamma x_0}{x_0 \Omega'}\right) \sin \Omega' t\right]$ and demonstrate by direct calculation that $x(0) = x_0$ and $\dot{x}(0) = v_0$.

1.8 Forced Oscillator With Damping

1-18. Show by direct substitution that $x = r^{-1} f \cos(\omega t - \theta)$ satisfies the forced damped oscillator equation of motion

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \cos \omega t$$

and that r and θ are the same as in (1.122) and (1.123).

1-19. An electric motor of mass 100 kg is supported by vertical springs which compress by 1 mm when the motor is installed. If the motor's armature is not properly balanced, for what revolutions/minute would a resonance occur?

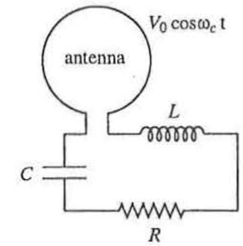
1-20. Find the initial conditions such that an underdamped harmonic oscillator will immediately begin steady-state motion under the time-dependent force $F = mf \cos \omega t$.

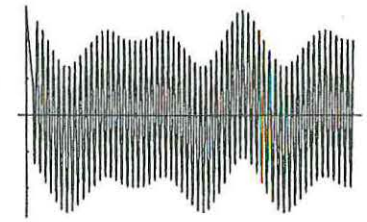
1-21. Find the steady-state solution for a damped harmonic oscillator driven by the force

$$F(t) = mf \sin \omega t$$

1-22. An AM radio station transmits a signal consisting of a carrier wave at frequency $\nu_c = 10^6$ Hz whose amplitude is modulated at frequency $\nu_m = 10^4$ Hz, as illustrated in the accompanying figure.

This means that many oscillations of the carrier occur while the modulation only changes slightly. A rudimentary radio receiver circuit is shown schematically in the accompanying figure. The incident radio waves induce an oscillating voltage $V_0 \cos \omega_c t$ in the antenna, where $\omega_c = 2\pi \nu_c$ and $V_0 \simeq 1$ mV.





a) Given the capacitance $C = 300$ pico-farads and resistance $R = 5$ ohms find the proper inductance L to create a resonance with the incident wave. *Hint: at first assume the resistance has little effect on the resonant frequency and then verify that this is a good approximation.*

b) Compute the damping constant and verify that the transients die out much faster than the modulation varies. This insures that the receiver will faithfully amplify the incident signal.

c) Compute the maximum voltage across the capacitor in terms of the above value of V_0. This is the voltage amplification of the circuit.

d) An adjacent station in carrier frequency is 20 kHz higher. If our receiver is tuned to the original 10^6 Hz how much will the adjacent stations's carrier be amplified?

1-23. A critically damped oscillator with $\omega_0 = 1$ rad/sec is acted upon by a driving force F_{driver}

a) Find a particular solution for $F_{\text{driver}} = mfe^{t}$.

b) Find a particular solution for $F_{\text{driver}} = mfe^{-t}$. *Hint: Try* $x = At^n e^{-t}$ *for* $n = 0, 1, 2$.

c) Using the preceding results obtain the general solution for $F_{\text{driver}} = mf \cosh t$ with initial conditions $x(0) = \dot{x}(0) = 0$.

1-24. An undamped harmonic oscillator with natural frequency ω_0 is subjected to a driving force $F(t) = ae^{-bt}$. The oscillator starts from rest at the origin ($x = 0$) at time $t = 0$. Find the solution of the equation of motion which satisfies the specified initial condition.

1-25. Find the average power dissipated per driving period by the frictional force of a sinusoidally driven harmonic oscillator in steady state. (Recall that power = force × velocity.) Show that maximum dissipation occurs at $\omega = \omega_0$ and evaluate this maximum.

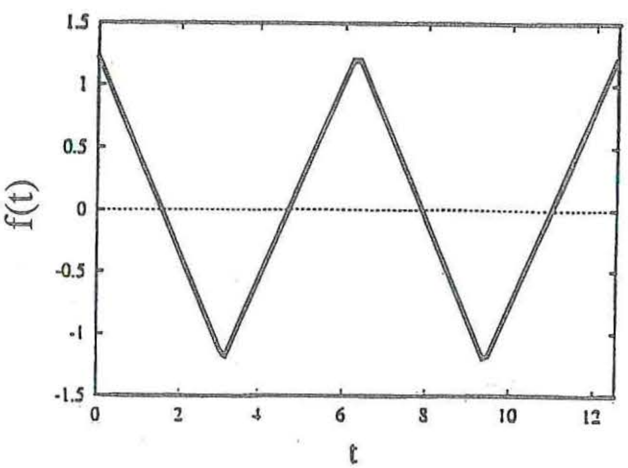
1-26. A sawtooth wave (see accompanying figure) can be decomposed into an infinite sum of cosines

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(\omega_n t),$$

where $\omega_n = (2n+1)\omega$ and ω is the "angular frequency" of the sawtooth. Find the steady-state motion of an oscillator driven by this force per unit mass

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f(t).$$

Hint: find the solution for a given ω_n and use the principle of superposition.



Chapter 2

ENERGY CONSERVATION

There are three important conservation laws of mechanics: energy, momentum, and angular momentum. The three laws can be derived from Newtonian theory. However, their range of validity is much broader, extending even to the domain of relativistic elementary particles, although slightly changed in form. In their ramifications in all branches of science, these conservation laws have exceptionally far-reaching consequences. In this chapter we discuss energy conservation and then in later chapters we take up in turn momentum and angular momentum conservation.

2.1 Potential Energy

To derive the energy-conservation law in the case of one-dimensional motion, we start with the second law of motion for a body of mass m

$$\frac{d}{dt}(mv) = F(x, v, t) \tag{2.1}$$

and multiply by v. Since $v\,dv/dt = \frac{1}{2}d(v^2)/dt$ we obtain the equation

$$\frac{d}{dt}\left(\tfrac{1}{2}mv^2\right) = F(x, v, t)v \tag{2.2}$$

Substituting $v = dx/dt$ on the right-hand side and integrating with respect to t gives

$$\tfrac{1}{2}mv^2(t_2) - \tfrac{1}{2}mv^2(t_1) = \int_{t_1}^{t_2} F(x(t), v(t), t)\frac{dx}{dt}dt = \int_{x_1}^{x_2} F(x, v(x), t(x))dx \tag{2.3}$$

The left-hand side is the difference at two times of the familiar expression for the kinetic energy

$$K = \tfrac{1}{2}mv^2 \tag{2.4}$$

Equation (2.3) is the *Work-Energy theorem*

$$K_2 - K_1 = \Delta K = \text{Work} = \int_{x_1}^{x_2} F(x, v(x), t(x))\,dx \tag{2.5}$$